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Self-Projective Rational Sextics.[*]

By R. M. Winger.

Introduction. General Considerations.

1. Elsewhere[†] the various types of self-projective rational quartic and quintic curves have been tabulated. In this connection two curves are said to be of different *type* if they are invariant under different groups, or if, when invariant under the same group, they are projectively distinct, and one is not a special case of the other, in the sense that one can not be obtained by continuous variation of the arbitrary constants in the parametric equations of the other. In the present paper a complete classification of self-projective ternary rational sextics[‡] is undertaken, and some of the more immediate properties are inferred.

The study is interesting, not only as illustrating a fruitful method of investigation, but also because it parallels the theories of binary and ternary collineation groups. In particular, several classical configurations associated with the curves are brought to light.

Since we are dealing with parallel theories some conventions of language are convenient. Thus, we restrict *involution* to the binary domain, *i. e.*, the parameter. We may, however, use "point" in place of the awkward "parameter of a point" when the meaning is clear. Binary and ternary groups of order n are designated respectively by g_n and G_n, and the corresponding curve by ρ_n^6. An involutory ternary collineation is called invariably a *reflexion.*

2. It is essential to bear in mind the relation between the binary and ternary groups. Thus, any rational curve is invariant under the general binary linear transformation of the parameter which is a continuous three-parameter

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[*] Read before the American Mathematical Society, September 9, 1913.

[†] American Journal of Mathematics, Vol. XXXVI, January, 1914. This paper is referred to in the sequel as "D." It is believed that all types of quartics are obtained there, but the list of quintics would have to be supplemented to conform to the present criterion. Thus the curve (foot-note, p. 68) should be included as a distinct type. There is also a second $\rho_3{}^5$, $x_0 = t^5 + t^2$, $x_1 = t^3$, $x_2 = 1$.

[‡] This implies a classification of self-projective rational sextics in S_3, since the ternary sextic projectively defines a quaternary sextic, the line sections of the one and the plane sections of the other being apolar binary forms. The problem for the general plane sextic is solved by A. H. Tappan, American Journal of Mathematics, Vol. XXXVII, July, 1915.

group, but the effect may be regarded merely as a change of the *naming* of the points. Hence, in general, the curve is not invariant under the ternary group so generated. *If, however, a binary g_n permutes the ∞^2 line sections, the curve admits a ternary G_n and is self-projective.*

Again, *if a rational curve admits a ternary G_n, this G_n on the points is effected by a binary group on the parameter of order n and not less,* else the curve would be a locus of fixed points under some elements of G_n, which is impossible, since there is only a finite number of such points.

A cyclic g_n has just two fixed points, and at these points the directions (tangents) are also fixed. The corresponding G_n has the same fixed points and directions, but it may have other fixed points on the curve, which, therefore, must be multiple points of order n.

There is a single type of cyclic g_n, viz.,

$$t' = \varepsilon t, \quad \varepsilon = e^{\frac{2\pi i}{n}}.$$

But the ternary cyclic groups separate broadly into two classes: (1) the homologies with multipliers 1, 1, ε^a with a center and axis, the one a point of fixed lines and the other a line of fixed points; and (2) those with a fixed triangle and multipliers 1, ε, ε^b $(a=1, \ldots, n-1;\ b=2, \ldots, n-1)$. These latter divide into types projectively distinct as n becomes larger.

The following principles are very useful in studying rational curves admitting a reflexion. A fixed parameter of an involution can lie only at the center or on the axis of reflexion. All other intersections of the axis or of lines on the center must be conjugate pairs of the involution. Hence, *the intersections of the axis are either contacts, simple or of higher order, of tangents from the center, or multiple points. And the tangents from the center either have their contacts on the axis or are multiple tangents.*

3. The accompanying table exhibits in canonical form the parametric equations of the various types of self-projective rational sextics, together with the generating transformations, binary and ternary, of their characteristic groups. For the dihedral groups only transformations for the corresponding invariant cyclic subgroups are given. They are to be combined in every case with $t'=1/t$, $x_0'=x_1$, $x_1'=x_0$, $x_2'=x_2$. Cyclic, dihedral, tetrahedral, octahedral and icosahedral groups are designated respectively by C, D, T, O and I; u stands for the substitution $t=\frac{t'+i}{t'-i}$, while v denotes the transformation T of Klein;[*] $\omega^3=i^4=\varepsilon^5=1$. For the tetrahedral sextic,

$$r=\frac{16a}{4-a^2}, \quad s=\frac{3a^2+20}{a^2-4}.$$

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[*] "Ikosaeder," p. 41.

Following the table, the trilinear equations of some of the curves are given.

	Transformations.				*Equations.*			
	t'	x_0'	x_1'	x_2'	x_0	x_1	x_2	
C_2	$-t$	x_0	x_1	$-x_2$	$t^6+at^4+bt^2$	ct^4+dt^2+1	t^5+et^3+t	(1)
C_2	$-t$	$-x_0$	$-x_1$	x_2	t^5+at^3	bt^3+t	$t^6+ct^4+dt^2+1$	(2)
C_3	ωt	ωx_0	$\omega^2 x_1$	x_2	t^4+at	bt^5+t^2	t^6+ct^3+1	(3)
C_3	ωt	x_0	x_1	ωx_2	t^6+at^3	bt^3+1	t^4+t	(4)
C_3	ωt	ωx_0	ωx_1	x_2	t	t^4	t^6+at^3+1	(5)
C_4	it	$-x_0$	x_1	ix_2	t^6+at^2	bt^4+1	t^5+t	(6)
C_4	it	$-x_0$	x_1	$-ix_2$	t^6	t^4+1	t^3	(7)
C_5	εt	x_0	εx_1	$\varepsilon^4 x_2$	t^5+1	t^6+at	t^4	(8)
C_5	εt	$\varepsilon^3 x_0$	εx_1	x_2	t^3	t^6	t^5+1	(9)
C_5	εt	εx_0	εx_1	x_2	t	t^6	t^5+1	(10)
C_6	$-\omega^2 t$	$-\omega^2 x_0$	$-x_1$	x_2	t	t^3	t^6+1	(11)
C_6	$-\omega^2 t$	ωx_0	$-x_1$	x_2	t^2	t^3	t^6+1	(12)
C_6	$-\omega^2 t$	$-\omega^2 x_0$	$\omega^2 x_1$	x_2	t	t^4	t^6+1	(13)
C_6	$-\omega^2 t$	$-\omega^2 x_0$	ωx_1	x_2	t	t^2	t^6+1	(14)
C_∞	αt	$\alpha^6 x_0$	αx_1	x_2	t^6	t	1	(15)
D_4	$-t$	x_0	x_1	$-x_2$	$t^6+at^4+bt^2$	bt^4+at^2+1	t^5+et^3+t	(16)
D_4	$-t$	$-x_0$	$-x_1$	x_2	t^5+at^3	at^3+t	$t^6+bt^4+bt^2+1$	(17)
D_6	ωt	ωx_0	$\omega^2 x_1$	x_2	t^4+at	at^5+t^2	t^6+bt^3+1	(18)
D_8	it	$-x_0$	x_1	ix_2	t^6+at^2	at^4+1	t^5+t	(19)
D_8	it	$-x_0$	x_1	$-ix_2$	t^6+at^2	at^4+1	t^3	(20)
D_{10}	εt	εx_0	x_1	$\varepsilon^3 x_2$	t^6+at	at^5+1	t^3	(21)
D_{12}	$-\omega^2 t$	$-\omega^2 x_0$	ωx_1	x_2	t	t^5	t^6+1	(22)
T_{12}	d_4, u	D_4 (16), U			$t^6+rt^4+st^2$	st^4+rt^2+1	t^5+at^3+t	(23)
O_{24}	d_8, u	D_8 (19), U			t^6-5t^2	$-5t^4+1$	t^5+t	(24)
O_{24}	d_8, u	D_8 (20), U			t^6+3t^2	$3t^4+1$	t^3	(25)
I_{60}	d_{10}, v	D_{10}, V			t^6+3t	$3t^5-1$	$-5t^3$	(26)

(7) $(x_2^2-x_0x_1)^3+x_0^4x_2^2=0,$ (9) $(x_0^2-x_1x_2)^3+x_0x_1^5=0,$

(11) $(x_0^3+x_1^3)^2-x_0^3x_1x_2^2=0,$ (12) $x_0^6+x_1^6-x_0^3x_1^2x_2=0,$

(13) $(x_0^2+x_1^2)^3-x_0^2x_1x_2^3=0,$ (14) $x_0^6+x_1^6-x_0^4x_1x_2=0,$

(22) $(x_0^3-x_1^3)^2+x_0x_1x_2^2(4x_0x_1-x_2^2)=0.$

I. *The Cyclic Sextics.*

4. The curves invariant under the cyclic groups are referred uniformly to a fixed triangle, when the equations appear in the canonical form.

Curve (1): x_2 is the axis of reflexion and cuts out the fixed points of the involution and two double points. Four double lines meet at the center ξ_2.*

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* The vertex of the triangle of reference opposite x_i will be denoted by ξ_i.

5. Curve (2): There is a biflecnode at ξ_2, the center of reflexion, whose parameters are the fixed points of the involution. x_2 is the axis and cuts out three double points. The intersections of the axis being completely accounted for, two double lines meet at the center.

The remaining six double points must lie in pairs on lines through the center, *i. e., six of the double points are harmonically perspective from a seventh.*

6. Curve (3) has a double point at ξ_2 whose parameters are fixed under the binary g_3. x_0 and x_1 are the nodal tangents.

7. Curve (4): ξ_2 is the center and x_2 the axis of the homology. x_2 cuts out a point of inflexion and a cusp, whose parameters are the fixed points of the g_3. The remaining intersections of x_2 being fixed points of G_3, in consequence of a theorem of § 2, must constitute a triple point.

8. Curve (5): The center of the homology is a multiple point arising from the junction of a cusp of higher order with an undulation, equivalent to one cusp, three double points, six flexes and nine double lines. The parameters are the fixed points of g_3. The intersections of the axis x_2 are fixed points of G_3 and must constitute, therefore, two triple points. The remaining six flexes lie in threes on lines through the center.

9. The cyclic G_5: Since the curve is of order 6 and the group of order 5, one line section must be of the form (1) t^6+at. The only other binomial line section is (2) t^5+b. The remaining possibilities are (3) t^2, t^3, t^4. Inasmuch as sections (1) and (3) contain a common factor, (2) must always be used. We have then three types of cyclic ρ^6_5, the first two belonging to the same group, the third to an homology.

10. Curve (8) has a "touching undulation tangent,"* x_2, which consumes two flexes and four double lines, and whose contacts are the fixed points of the binary group.

The six flexes of curve (9) lie on a conic whose remaining intersections fall at the triple point ξ_2.

11. Curve (10) is the sextic case of the rational ρ^n_{n-1}, $x_0=t$, $x_1=t^n$, $x_2=t^{n-1}+1$, or $x_0(x_0+x_1)^{n-1}-x_1x_2^{n-1}=0$. ξ_2 is the center and x_2 the axis of the homology. The curve has an $(n-1)$-fold point on x_2. x_0 and x_1 are tangents with $(n-1)$-point and n-point contacts, respectively, consuming $(2n-5)$ flexes and $(n-3)^2$ double lines. There remain $(n-1)$ flexes and $(n-1)(n-3)$ double lines. The flexes lie on a line through the center.

12. The four varieties of ρ^6_6 are found at once from the formula, "D.," § 10.†

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* *i. e.*, an undulation tangent with an additional (simple) contact.

† $x_0=t^r,\ x_1=t^s,\ x_2=t^6+1,\ r\neq s,\ 1\leqq\left\{\begin{matrix}r\\ s\end{matrix}\right\}\leqq 5.$

The binary group contains as subgroups a g_3 and a g_2; *i. e.*, the curves admit a reflexion. Note, however, that there are two types of ternary G_6, the first three curves belonging to one, and the fourth to the other. These types may be characterized, the one by the fact that its subgroup of order 3 is an homology, the other as possessing a pencil of proper fixed conics.

13. Curve (11) has a multiple point at ξ_2 which is equivalent to two cusps, five double points, one double line and two flexes, consuming in all six flexes and twelve double lines. There are, however, but two distinct parameters, the fixed points of the binary group.

Sets of conjugate points are cut out by the line pairs $x_2^2+\lambda x_1^2=0$. For example, for particular λ there are the following pairs: (a) one on three other double points, $\lambda=0$; (b) one on six other flexes, $5\lambda=16$; (c) one which is a pair of triple tangents, $\lambda=-4$.

14. Curve (12) has a fivefold point at ξ_2, due to a cusp falling at a special triple point. The multiple point is equivalent to three cusps and seven double points, and consumes therefore six flexes and fifteen double lines. The two parameters are fixed under the binary group. The cuspidal tangent, x_1, is the axis of reflexion, the complete intersections falling at ξ_2. x_0 counts as a simple tangent from ξ_1, the center of reflexion. The other six tangents, the curve being of class 7, must consist of three double lines.

As before, sets of conjugate points are cut out by the line pairs $x_2^2+\lambda x_1^2=0$. The following pairs are of interest: (a) one on the six flexes, $2\lambda=1$; (b) one pair of triple lines, $\lambda=-4$; (c) one on six contacts of the three double lines, $2\lambda=-9$. All singularities of the curve are now accounted for, the three triple tangents counting for six double lines.

15. Curve (13): This is obtained from (5), by placing $a=0$, which amounts to imposing a reflexion. The curve has the same singularities as before. x_0, the cuspidal tangent, is the axis of reflexion, all intersections falling at ξ_2. x_1 counts three times as a tangent from ξ_0, the center of reflexion, the other six tangents forming double lines.

Sets of conjugate points are cut out by the line pairs $x_0^2+\lambda x_1^2=0$, six of the intersections having been taken up at ξ_2. For varying λ the following special pairs are to be noted, one on (a) two triple points, $\lambda=1$; (b) contacts of three double lines, $2\lambda=-1$; (c) six flexes, $2\lambda=5$; (d) six contacts of tangents from ξ_1, $5\lambda=-1$.

16. Curve (14) has a fivefold point at ξ_2, equivalent to three cusps and seven double points, caused, however, by a simple branch passing through a special fourfold point whose four parameters coincide. ξ_0 is the center and

x_0 the axis of reflexion. The six intersections of the axis lie at the multiple point. x_1 counts once as a tangent from ξ_0, the other six tangents being three double lines.

Sets of six conjugate points are cut out by the pencil of double-contact conics $x_0^2+\lambda x_1 x_2=0$, six of the intersections falling at the multiple point. Of these conics may be mentioned: (a) one on the six flexes; (b) three each on six contacts of double lines.

17. If ρ^6 admits a cyclic G_7, it admits an infinite group.[*] There is just one type, (15). There is a hyperosculation point at 0 and the dual singularity at ∞, the two consuming all of the singularities of the curve.

II. *The Dihedral Sextics.*

18. Closely associated with dihedral groups is the question of symmetry.[†] The group method is particularly effective in discussing the invariant curves, the properties of which are of unusual interest.

Curve (16): The fixed parameters of the three involutions are cut out by the axes of reflexion and name contacts of tangents from the centers. The other intersections of the axes must be interchanged under the involutions; that is, *two double points lie on each axis.* The remaining four double points are a set of four under the ternary group. Since the intersections of the axes are completely accounted for, the other tangents from the centers must be double tangents. Hence, *four double lines meet at each center of reflexion.*

19. Curve (17): A second vertex of the triangle of centers is a biflecnode and the properties of ρ_2^6, (2), are accordingly repeated. The third center, however, is off the curve, and the fixed parameters of the corresponding involution are cut out by the third axis, naming contacts of tangents from the center. Hence, *four double lines meet at this center and the fixed points of all three involutions lie on its axis.*

The remaining double points are a set of four under G_4. We may say, then: It happens twice that six double points are harmonically perspective from a seventh, while from the third center run two lines each carrying two double points and two lines carrying three each.

20. Curve (18): The three axes of reflexion meet at a double point, whose parameters, we saw (§ 3), are fixed under the cyclic g_3. The fixed points of the three involutions lie in pairs on the axes and are contacts of tangents from the centers. As above, it follows that *a second double point lies on each axis. The other six double points are a general set and lie on one of the double-contact conics* $x_0 x_1+\lambda x_2^2=0$, which in general cut out two sets of conjugate points.

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[*] "D.", § 10.

[†] "D.," § 4.

All intersections of the axes are accounted for. Hence, *four double lines meet at each center.*

The flexes must constitute two sets of six, which lie on two of the invariant conics.

21. Curve (19): The only special sets of parameters are given by t, t^4+1 and t^4-1, the first two being cut out by x_2. The two pairs t^4-1 lie on the axes $x_0^2-x_1^2=0$, being contacts of tangents from the corresponding centers. The residual intersections of these axes are pairs of double points. Since there are no other fixed parameters, all the intersections of the axes $x_0^2+x_1^2=0$ must be interchanged by their respective involutions. That is, *three double points lie on each of these axes.*

Now the cyclic G_4 contains a reflexion, whose center is the intersection, ξ_2, of the four axes above and whose axis, x_2, is their line of centers. The fixed parameters of the involution are 0 and ∞, while t^4+1 names two double points. The flex form* is found to be $(t^4+1)\{at^8+(3a^2-16a+15)t^4+a\}$. Hence, *the two double points on x_2 are biflecnodes.*

The biflecnodes are centers of reflexion, the parameters being fixed points of the corresponding involutions. The four tangents from each must make up two double lines, since all of the intersections of the axes are double points; while four double lines meet at each of the other three centers. This accounts for sixteen of the twenty-four double lines.

General conjugate sets of eight points are cut out by the system of invariant conics, all of which have contact with ρ^6 at 0 and ∞. Of these may be mentioned: (a) one on eight flexes; (b) one on eight flex tangents; (c) one with four additional contacts; (d) two each on four double points; (e) two each on eight tangents at double points; (f) three each on four double lines; (g) three each on eight contacts of double lines. (c) is a perspective conic.

Many interesting special cases arise for particular values of a. Among these are roots of the discriminant $(a-1)(a-3)(a-5)(3a-5)$ of the flex form, exclusive of t^4+1, whose vanishing, owing to the symmetry, means four pairs of equal roots. To these values of a correspond the following curves: (1) $a=1$, degenerate; (2) $a=3$, cusps at t^4-1; (3) $a=5$, two double points, each with two five-point contact tangents; (4) $3a=5$, undulations at t^4-1. (2) is self-dual, having four double points, four double lines, two double-cusp tangents and two biflecnodes.

22. Curve (20): The line x_2 is a double-cusp tangent, which counts for a double line, besides consuming eleven double lines and four flexes. Four double lines meet at the center ξ_2.

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* J. I. Tracey, *Johns Hopkins University Circular,* July, 1913, has developed the flex form for the general rational plane curve.

On each of the other four axes lie two double points and two contacts of tangents from the centers, the fixed points of the corresponding involutions; and at each of the four centers two double lines meet, accounting for the remaining double lines.

All other sets of conjugate points are sets of eight, and are cut out by the pencil of invariant conics, which pass through the cusps. Of these conics we note: (1) one on the eight flexes; (2) one on eight flex tangents; (3) two each with four contacts besides the cusps; (4) two each on four double points, *i. e.*, on six double points; (5) two each on eight tangents at double points; (6) two each on four double lines; (7) two each on eight contacts of double lines.

The flex form is $3at^8+(a^2-27)t^4+3a$. Corresponding to the factors of the discriminant are two pairs of projectively equivalent curves, characterized as follows: (1) $a=3$, cusps at t^4-1; (1′) $a=-3$, cusps at t^4+1; (2) $a=9$, undulations at t^4+1; (2′) $a=-9$, undulations at t^4-1.

23. Curve (21): The line x_2 is a double-flex tangent, which counts for two flex tangents and four double lines, the contacts being fixed points of the cyclic g_5.

The fixed points of the five involutions lie in pairs on the five axes and are contacts of tangents from the five centers. The other intersections of the axes are double points, and the other tangents from the centers are double lines. That is, *two double points lie on each axis and two double lines meet at each center.*

The double-flex tangent is the common chord of the pencil of invariant conics, the contacts being at the flexes. The remaining intersections of the conics are sets of ten conjugate points. Noteworthy among these conics are the following: (1) one on ten, *i. e.*, on all twelve flexes; (2) one on ten flex lines; (3) two each on five double points; (4) two each on ten tangents at double points; (5) two each with five contacts; (6) two each on five double lines; (7) two each on ten contacts of double lines. The equation of (1) is

$$3x_0x_1+(5a^2-30)x_2^2=0.$$

The factors of the discriminant of the flex form, exclusive of 0 and ∞, and the significance of their vanishing are: (1) $2a=3$, cusps at t^5-1; (1′) $2a=-3$, cusps at t^5+1; (2) $4a=9$, undulations at t^5+1; (2′) $4a=-9$, undulations at t^5-1. Corresponding to these values of a are two pairs of projectively equivalent curves.

24. Curve (22):* The vertex ξ_2 is a double point, the tangent at each branch of which has five-point contact, consuming six flexes and six double lines.

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* See "D.," § 10.

The nodal parameters are fixed under the cyclic g_6, the node itself being a fixed point of the complete ternary G_{12}. ξ_2 is also a center of reflexion with x_2 as axis.

The other six axes of reflexion constitute two non-conjugate sets of three lines,[*] $x_0^3-x_1^3=0$, $x_0^3+x_1^3=0$, meeting at the double point ξ_2. Designate these respectively as lines α and β and the corresponding centers by points a and b.

Of the special sets of parameters, t^6-1 lie in pairs on axes α and name contacts of tangents from centers a. A double point completes the intersections of each of these three axes. Accordingly, the remaining tangents from centers a are double tangents; in other words, *four double lines meet at each center a.*

The other special set of parameters, t^6+1, cut out by x_2, names three double points. Since all other parameters on the curve are sets of twelve, the six flexes must be one of these special sets of six. Hence, *the three double points on x_2 are biflecnodes,* one lying on each axis α and being a center b.

Again, all the intersections of each axis β must be interchanged by an involution. Hence, *the residual six double points lie in pairs on the axes β.* That is, *six double points are perspective from a seventh,* the axis of perspective being x_2, the line of three biflecnodes. In addition *they are triply perspective, the centers and axes of perspective being the centers a and the axes α of reflexion.*

From each biflecnode run three double lines, so that the aggregate of eighteen is now accounted for. Thus, on the line x_2 are three points, a, each carrying four double lines, and three biflecnodes, b, each carrying two double lines.

The invariant conics all touch the tangents of the special double point ξ_2 where they meet the line x_2, accounting thus for eight common lines. Of these conics, whose complete intersections are sets of conjugate points, we record: (1) one with six contacts, t^6-1; (2) one on six ordinary nodes and (3) one on twelve nodal tangents of the same; (4) one on the six flex tangents; (5) three each on six double lines; (6) three each on twelve contacts of double lines. (1) is perspective. All twenty common lines of each are accounted for except those of (2) and (6).

III. *The Tetrahedral, Octahedral and Icosahedral Sextics.*

25. The equations of the tetrahedral sextic (23) are found from those of the dihedral ρ_4^6, (16), by requiring that the quadratics naming the fixed points of the involutions, *i. e.*, that the axes of reflexion, be permuted cyclically by the transformation u.

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[*] Consult "D.," § 4.

The parameters of the curve in general occur in conjugate sets of twelve. But there are two special sets of four and one set of six, corresponding to the vertices of the two tetrahedra, T and T', and to the mid-points, O, of the edges.* The six points O are the fixed points of the involutions and lie in pairs on the axes of reflexion. The axes cut out, besides, six double points whose parameters are a general set of twelve. The remaining four double points then give rise to a set of eight. Since there is no set of eight, these points must be the two sets T and T'. Now, the points T are interchanged in pairs by the three involutions, and the same is true of the points T'. Hence, *there is one point of each set at each of the four double points; that is to say, the parameters of each double point are the fixed points of a cyclic* g_3. The effect of each of the ternary G_3's, therefore, is to leave one double point fixed, while it permutes cyclically the other three. Since all special sets of points are accounted for, the flexes are a general set of twelve. Four double lines meet at each center.

26. Curve (24): When $a=0$ or ∞ in the equations of the tetrahedral sextic, the curve admits the complete octahedral G_{24}. The binary group contains only three special sets of points fewer than twenty-four, viz., †

$$O,\ t(t^4-1)\,;\quad C,\ t^8+14t^4+1\,;\quad M,\ t^{12}-33t^8-33t^4+1.$$

Since the G_{24} contains three conjugate dihedral G_8's, anything true once for a ρ_8^6 will be true three times in general for the octahedral sextic, and the properties of the latter can be inferred readily from a consideration of the former, of which it is a special case. Thus, each G_8 possesses a pencil of invariant conics with contacts at the fixed points of an involution of the "four-group" g_4. *There is, however, a single fixed conic of* G_{24}. *Belonging to each system, it has double contact with each conic of the three conjugate systems and six contacts with the curve. These contacts, being a set of six, are the vertices of the octahedron, O.*

The flexes are a set of twelve parameters, and coincide therefore with the points M, as is verified at once from Tracey's equation. But we saw (§ 21) that four of the flexes of ρ_8^6, (19), lie at two double points. Hence, in this case, *all of the flexes fall at double points, giving rise to six biflecnodes. The remaining four double points are named by a set of eight parameters, which constitute therefore the cube vertices C.*

Now, each of the biflecnodes of ρ_8^6 lies on a line with a pair of other double points. And since the double points must be treated symmetrically, this will happen for every biflecnode. Hence, *the biflecnodes are the complete intersections of four lines, which form with the four ordinary double*

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* Klein, "Ikosaeder," Chapter II, § 11.

† Klein, "Ikosaeder," p. 54.

points a quadrangle-quadrilateral configuration,[*] a self-dual configuration such that each side of the quadrilateral is the polar of one vertex of the quadrangle with reference to the triangle of the other three.

The biflecnodes are centers of reflexion, from each of which run two double lines, while four double lines meet at the centers of reflexion of the four-group.

27. Curve (25): This is seen to be the *projective astroid*, the miscalled "hypocycloid of four cusps," since there are six. The group property might have been anticipated, since the curve is the exact dual of the projective emniscate, long known to admit the octahedral group.[†]

All of the flexes and twenty-one of the double lines are consumed in the formation of the cusps. There are three double-cusp tangents, which count as double lines. The only other singularities of the curve are four ordinary double points.

The cusps, whose parameters are the octahedron vertices, lie on a conic, the invariant conic of G_{24}. The other double points, being a set of eight, are the cube vertices. Each of the three conjugate sets of invariant conics has a double-cusp tangent as common chord with the contacts of the cusps. One conic of each set has four contacts with the curve. In other words, *there are three conics, each on two cusps, and having four contacts. These contacts constitute a set of twelve, and must therefore be the median points M.*

It may be remarked that the ternary octahedral group is the group of permutations on the four ordinary double points of both types of sextics invariant under it.

28. The icosahedral sextic is a curve of particular interest, not only on account of its striking and beautiful properties, but because it is the curve of lowest order other than a conic to admit the icosahedral group. It may be obtained from our ρ_{10}^6, (21), by imposing the further condition of invariance under the transformation T of Klein (p. 41).

Many properties of ρ_{60}^6 can be deduced from the dihedral sextics, of which it is a special case. Thus, we saw that ρ_{10}^6 has a singular line counting for two flex tangents and four double lines. Hence, *the icosahedral sextic has six double-flex tangents consuming all the flex and double lines.* Their equations are

$$x_2=0, \quad \varepsilon^i x_0+\varepsilon^{-i}x_1+x_2=0, \quad \varepsilon^5=1, \qquad i=0, \ldots, 4.$$

Again, the double-contact conics of a G_{10} have as common chord a multiple tangent with the two flexes as points of contact. Of these conics there are: (1) two each on five double points; (2) two each on ten tangents at these double points; (3) two each with five contacts; (4) one on five double-flex tangents, which make up the total of twenty common lines. (5) Among the invariant conics of a dihedral G_6 there is one on six double points. The dihedral groups being conjugate, theorems (1)–(4) are true six times, theorem (5) ten times.

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[*] Veblen and Young, "Projective Geometry," Vol. I, p. 44.

[†] Berzolari, *Istituto Lombardo Rendiconti*, Series II, Vol. XXXVII, pp. 277 and 304.

*The flexes, being a set of twelve, are the icosahedron vertices I. The double points are a set of twenty parameters, which are therefore the dodecahedron vertices H. The fixed points of the fifteen involutions lie in pairs on the fifteen axes of reflexion, constituting a set of thirty, which are the mid-points of the edges, J.**

29. We saw ("D.," § 4) that every ternary dihedral group has a fixed line, the line of centers, and a fixed point, the intersection of the axes of reflexion. The dihedral G_4, however, containing only reflexions, has three such points and lines, forming a triangle. Thus, there are associated with the G_{60}, by means of the dihedral subgroups, six points, say a, ten points b and fifteen points c, and the corresponding lines α, β, γ. To these we now direct our attention.

In the first place, these points and lines are poles and polars with respect to the systems of double-contact conics set up by the groups ("D.," § 4). But *there is a single conic, N, invariant under the whole group, and belonging, therefore, to each system, namely the conic on the twelve flexes:* $x_0x_1+x_2^2=0$. Hence, the points and lines belong to the polar system of the conic N.

The points c and lines γ are the centers and axes of the fifteen reflexions. Lines γ meet by fives at points a, by threes at points b, and in pairs at points c, which accounts for the total of 105 intersections. A dual statement holds for points c. Now, the line of centers of each dihedral G_{10} is a double-flex tangent. That is, *the multiple lines are the lines* α. Again, we saw (§ 20) that the fixed point of the ternary dihedral G_6 is a double point of ρ^6. Hence, *the ten double points are the points b.* From each center of reflexion run two simple tangents, which have their contacts on the corresponding axis. There can be no other simple tangents and, the curve being of class 10, two double-flex tangents meet at each center. That is, *the fifteen centers of reflexion are the complete intersections of the six singular lines.*

To sum up, *the complete junctions of the six points a are the axes of reflexion* γ *which meet by threes in the ten double points b. The lines* α *are the double-flex tangents. Their fifteen intersections are the centers of reflexion c, which lie by threes on lines* β. *In other words, the points and lines define the dual configuration of a tenfold Brianchon six-point and a tenfold Pascal hexagram.*

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* These invariant forms are given, Klein, p. 56.